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# Some Remarks on Regular\* Semigroups (半群とその周辺)

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SOME REMARKS ON REGULAR  $*$  SEMIGROUPS

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A semigroup  $S$  is called to be fundamental if its only one congruence contained in the Green's relation  $\mathcal{H}$  on  $S$  is the trivial one. In his paper [2], Hall gives us the construction of a fundamental regular semigroup which is the generalization of [1] and [3]. In this paper, we shall study a fundamental regular  $*$  semigroup.

A semigroup  $S$  with a unary operation  $*$ :  $S \rightarrow S$  is called a  $*$  semigroup if it satisfies

- (i)  $(x^*)^* = x,$
- (ii)  $(xy)^* = y^*x^*.$

Let  $S$  and  $T$  be  $*$  semigroups. A mapping  $\phi: S \rightarrow T$  is called a  $*$  homomorphism if  $\phi$  is a (semigroup) homomorphism and  $x^*\phi = (x\phi)^*$  for all  $x$  in  $S$ . A relation  $\nu$  on  $S$  is called a  $*$  relation on  $S$  if  $(x, y) \in \nu$  implies  $(x^*, y^*) \in \nu$ . A  $*$  semigroup  $S$  is called a regular  $*$  semigroup if it satisfies

- (iii)  $xx^*x = x.$

An idempotent  $e$  in  $S$  such that  $e^* = e$  is called a projection.

The following result due to Nordahl and Scheiblich is an important property of a regular  $*$  semigroup.

RESULT 1 ([4]). Let  $S$  be a regular  $*$  semigroup. Then each  $\mathcal{L}$ -class and each  $\mathcal{R}$ -class in  $S$  contain one and only one projection. Let  $e$  and  $f$  be projections in  $S$ . Then  $ef$  is an idempotent in  $S$ .

Hereafter, a regular  $*$  semigroup  $S(P)$  means that  $S$  is a

regular  $*$  semigroup with the set of projections  $P$ .

LEMMA 2. Let  $S(P)$  be a regular  $*$  semigroup, and let  $E$  be the set of idempotents in  $S$ . Then  $E = P^2$ . More precisely, for any idempotent  $e$ , there exist projections  $f$  and  $g$  such that  $e \mathcal{R} f$ ,  $e \mathcal{L} g$  and  $e = fg$ .

LEMMA 3. Let  $S$  be a regular  $*$  semigroup. For any element  $a$  and any projection  $e$ ,  $a*ea$  is also a projection.

A  $[*]$  congruence  $\nu$  on a regular  $[*]$  semigroup  $S$  is called an idempotent-separating  $[*]$  congruence if  $\nu \in \mathcal{H}$ .

THEOREM 4. Let  $\mu$  [ $\mu'$ ] be the maximum idempotent-separating  $[*]$  congruence on a regular  $*$  semigroup  $S(P)$ . Then  $\mu = \mu' = \{(a,b) \in S \times S: a*ea = b*eb \text{ and } aea* = beb* \text{ for all } e \in P\}$ .

Let  $S(P)$  be a regular  $*$  semigroup. For any element  $a$  in  $S$ , let  $\rho_a$  and  $\lambda_a$  be mappings of  $P$  into  $P$  defined by

$$e\rho_a = a*ea,$$

$$e\lambda_a = aea*.$$

It is clear that  $\rho_{ab} = \rho_a\rho_b$  and  $\lambda_{ab} = \lambda_b\lambda_a$ . Let  $A, B$  be subsets of  $P$ . A mapping  $\alpha: A \rightarrow B$  is called a partial isomorphism if  $\alpha$  is bijective and for  $a_1, a_2, \dots, a_n$  in  $A$ ,

$a_1a_2\dots a_n \in A$  implies  $(a_1\alpha)(a_2\alpha)\dots(a_n\alpha) \in B$  and  $(a_1a_2\dots a_n)\alpha = (a_1\alpha)(a_2\alpha)\dots(a_n\alpha)$ . If there exists a partial isomorphism

$\alpha: A \rightarrow B$ , we say  $A$  is partial isomorphic to  $B$ , and denote it by  $A \mathbb{D} B$ . For each  $e$  in  $P$ , let  $\langle e \rangle = \{f \in P: f \leq e\} = ePe$ .

Let  $\mathcal{U} = \{(e,f) \in P \times P: \langle e \rangle \mathbb{D} \langle f \rangle\}$  and for each  $(e,f) \in \mathcal{U}$ , let  $T_{e,f}$  be the set of all partial isomorphisms of  $\langle e \rangle$  onto  $\langle f \rangle$ . Let

$T_P = \bigcup_{(e,f) \in \mathcal{U}} \{(\rho_e\alpha, \lambda_f\alpha^{-1}): \alpha \in T_{e,f}\}$ . For convenience, we shall

denote  $(\rho_e \alpha, \lambda_f \alpha^{-1})$  simply by  $\phi(\alpha)$ . It is clear that  $T_P \subset \mathcal{T}_P \times \mathcal{T}_P^*$ , where  $\mathcal{T}_P^*$  is the dual semigroup of  $\mathcal{T}_P$ .

THEOREM 5. (i) Define a unary operation  $*$ :  $T_P \rightarrow T_P$  by  $\phi(\alpha)^* = \phi(\alpha^{-1})$ . Then  $T_P$  is a regular  $*$  subsemigroup of  $\mathcal{T}_P \times \mathcal{T}_P^*$ . Moreover, the set of projections of  $T_P$  is  $\{(\rho_e, \lambda_e) : e \in P\}$ , and it is partial isomorphic to  $P$ .

(ii) For each  $a$  in  $S$ ,  $(\rho_a, \lambda_a)$  is an element of  $T_P$ . Let  $\xi$  be a mapping of  $S$  into  $T_P$  defined by

$$a\xi = (\rho_a, \lambda_a).$$

Then  $\xi$  is a homomorphism whose kernel is the maximum idempotent-separating congruence on  $S$ .

(iii) For any  $(e, f)$  in  $\mathcal{U}$ ,  $\alpha \in T_{e, f}$  and  $g \in P$ ,

$$\phi(\alpha)^*(\rho_g, \lambda_g)\phi(\alpha) = (\rho_{(ege)\alpha}, \lambda_{(ege)\alpha}).$$

(iv)  $T_P$  is a fundamental regular  $*$  semigroup.

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